Unsteady Boundary Layer Flows of an Electrically Conducting Fluid in the Presence of the Transverse Magnetic Field

Masakazu KATAGIRI

Department of Electrical Engineering, Faculty of Engineering

1. Introduction

Recently, the problem of the re-entry of missiles, and rockets into the atmospheric zone has been attracted by many investigators in view of the importance in aero space sciences. The flights which move at the high speed more than the sound speed are subjected to the enormous frictional drags at the re-entry into the astmospheric zone, and the heat energy due to them is sufficient to be burnt down the body of the flights. This is undesirable in the development of aero space sciences. The several methods which put away this difficulty have been devised, as one of which there is the method, what is called, the magnetic brake, or the boundary layer control. This method uses the electromagnetic effects of the electrically conducting fluid. When the flights move at the high speed in the air, there occurs the shock wave near the front nose of the body and behind the shock wave the temperature of the air becomes to be so high that the air is slightly ionized, and have the electrically conducting property. If we apply the magnetic field to the flow field in some suitable ways, the motion of the body is expected to change due to the electromagnetic effects of an electrically conducting fluid, that is, the slightly ionized air, and the force exerted the body and the heating state of the body are expected to be able to control.

The magnetic brake, or the boundary layer control, have been investigated theoretically by Rossow[1], Neuringer and Mcllroy,[2] and others. They treated mainly the steady, two dimensional boundary layer flow of a viscous, electrically conducting fluid. Moreover, in order to simplify the problems they assume that the fluid is incompressible. This assumption enables us to analize mathematically. Some useful conclusions are obtained, that is, the drag which decelerate the motion of the body increases and the heat transfer rate which is the rate of the heat flow from a boundary layer degreases due to the electromagnetic effects of an electrically conducting fluid in the presence of the magnetic field.

In this paper, we do not intend to obtain any useful conclusions in aero space sciences. We rather intend to investigate these problems in the academic view-point. Generally, the fundamental equations governing the electromagnetic effects of an electrically conducting fluid are very complicated in the form. They consist of the modified Navier-Stokes' equations and Maxwell's electromagnetic field equations. They have more non-linear terms than in the equations of ordinary hydrodynamics. Nevertheless, some exact solutions are obtained, and some approximations are made use of in the branches, that is slow motion, boundary layer flow, slender body theory, and others. Approximations are analogous to that of ordinary hydrodynamics. These approximations fairy succeeded for the electrically conducting fluid. In this point of view, we treat the unsteady two dimensional boundary layer flow of a viscous, incompressible, electrically conducting fluid making use of the approximation analogous to Blasius' approximation in ordinary boundary layer flow[3).

2. Fundamental Equations

The fundamental equations of an electrically conducting fluid in the presence of the magnetic field consist of the equation of continuity, the modified Navier-Stokes' equations due to the electromagnetic effects, the energy equation included Juelean dissipation energy, Maxwell's electromagnetic field equations, and some complementary relations. These fundamental equations are unified macroscopically by Alfvén[4) and others for a fully ionized gases. These macroscopic equations are available for the electrically conducting fluid in order to investigate the macroscopic motions. The following assumptions are admitted for the fully ionized gases, or for the electrically conducting fluid[5) : the electric displacement current, and the convection current are negligible compared with the conduction current. Furthermore, for the electrically conducting fluid the electric field due to excess charges are neglected. In the incompressible case, the fundamental equations are as follows:

equation of continuity

$$\mathrm{div}\ \mathbf{V} = O, \tag{1}$$

modified Navier-Stokes' equations

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \mathrm{grad}\ \mathbf{V} = -\frac{1}{\rho}\ \mathrm{grad}\ p + \nu\ \nabla^2 \mathbf{V} + \frac{1}{\rho}\ \mathbf{j} \times \mathbf{B}, \tag{2}$$

where the electromagnetic effects are included as $\mathbf{j} \times \mathbf{B}$. In the incompressible case, all the flow quantities can be obtained independently of the energy equation, and then the energy equation is omitted here.

Maxwell's electromagnetic field equations

$$\mathrm{rot}\ \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = O\,, \tag{3}$$

$$\mathrm{rot}\ \mathbf{H} = \mathbf{j}\,, \tag{4}$$

$$\mathrm{div}\ \mathbf{D} = O\,, \tag{5}$$

$$\mathrm{div}\ \mathbf{B} = O. \tag{6}$$

The complementary relations are the equations of the conduction current, which is called the general Ohm's law.

$$\mathbf{j} = \sigma (\mathbf{E} + \mathbf{V} \times \mathbf{B}) . \tag{7}$$

Here, $\mathbf{V}$ is the velocity, ρ the density of the fluid, p the pressure, ν the kinematic viscosity, $\mathbf{j}$ the electric current density, $\mathbf{B}$ the density of the magnetic flux, $\mathbf{E}$ the electric field, $\mathbf{H}$ the magnetic field, $\mathbf{D}$ the density of the electric displacement and σ the electric conductivity. We adopt the rationalized M. K. S. units. There are the two following relations which connect $\mathbf{B}$ with $\mathbf{H}$, and $\mathbf{D}$ with $\mathbf{E}$, respectively:

$$\mathbf{B} = \mu \mathbf{H} , \tag{8}$$

$$\mathbf{D} = \varepsilon \mathbf{E} , \tag{9}$$

whers μ and ε are the magnetic permeability and the dielectric constant, respectively.

These equations are the fundamental equations for an electrically conducting fluid in the incompressible case. The investigations in the basis of the above equations are generally called "Magneto-hydrodynamics", or "Hydromagnetics".

3. Assumption for the Electromagnetic Field

For the simple flow, that is, Couette flow, Hagen-Poiseuille flow, etc., the exact solutions are obtained without any assumption for the electromagnetic field. For the electromagnetic field, the exact solutions under the suitable boundary conditions are obtained. However, for the rather complicated flows, that is, the boundary layer flow, the slow motion, etc., some assumptions for the electromagnetic field have to be imposed.

Now, we introduce the non-dimensional quantities defined by

$$\mathbf{V} = U\,\mathbf{V}', \quad \mathbf{r} = L\,\mathbf{r}', \quad p = \rho\, U^2\, p', \quad t = \frac{L}{U}\, t' ,$$

$$\mathbf{B} = B\,\mathbf{B}', \quad \mathbf{j} = J\,\mathbf{j}', \quad \mathbf{E} = U\,B\,\mathbf{E}',$$

where L is the reference length, U the reference velocity, B the reference magnetic flux density, and J the reference electric current density. (4) and (7) are represented in the non-dimensional forms

$$\mathrm{rot}'\ \mathbf{B}' = R_m (\mathbf{E}' + \mathbf{V}' \times \mathbf{B}') , \tag{10}$$

where R_m is defined by

$$R_m = \sigma \mu U L,$$

which is called "Magnetic Reynolds' number" analogous to the ordinary Reynolds' number. This non-dimensional parameter signifies the measure of the ionization, or of the conductivity. In many practical situations in aero space sciences, the air is slightly ionized, and the conductivity is very small. Therefore, we can assume that $R_m \ll 1$. Then, (10) becomes

$$\mathrm{rot}' \mathbf{B}' \fallingdotseq O. \tag{11}$$

From (11), we can write

$$\mathbf{B}' = \mathrm{grad}' \chi. \tag{12}$$

Putting (12) into (6), we obtain

$$\nabla'^2 \chi = O. \tag{13}$$

The magnetic field becomes to be static in spite of the motion of the fluid. (13) can be generally solved under the suitable boundary conditions. Particularly, if the body is immersed in the flow of an infinitely extending fluid, and if the uniform magnetic field is applied at an infinitely far distance, the solution of (13) with the boundary condition as above is constant in the whole field, when we assume that the magnetic permeabilities μ of the body and of the fluid are equal[6]. Therefore, the magnetic field induced by the interaction between the motion of the body and the applied magnetic field can be neglected as long as $R_m \ll 1$. We may consider only the applied magnetic field. The assumption $R_m \ll 1$ is called conveniently "magnetic Stoke's approximation". Rossow[1] investigated the two-dimensional, incompressible boundary layer flow under this assumption. Treating the unsteady two-dimensional, incompressible boundary layer flow, we assume $R_m \ll 1$ and neglect the induced magnetic field.

4. Boundary Layer Approximation

In a boundary layer flow, the plate or the surface of the body is laid in the x - z plane of the orthogonal coordinates, and the y-axis is chosen perpendicular to it. The flow of the free stream is in the x-direction. In the two-dimensional case, the components of the velocity and of the magnetic flux density are generally assumed to be

$$\mathbf{V} = (u, v, 0), \tag{14}$$

$$\mathbf{B} = (B_x, B_y, 0). \tag{15}$$

And all quantities are independent of the variable z. Under these assumptions we find that the electric current density has only z component from (4), and the electric field has also the z component only from (7). In the steady flow, the electric field reduces to have the constant magnitude from (3).

Putting (7) into (2), we have in the non-dimensional form

$$\frac{\partial \mathbf{V}'}{\partial t'} + \mathbf{V}' \cdot \mathrm{grad}' \mathbf{V}' = - \mathrm{grad}' p' + \frac{1}{R} \nabla'^2 \mathbf{V}' + SR_m (\mathbf{E}' + \mathbf{V}' \times \mathbf{B}') \times \mathbf{B}', \tag{16}$$

where R and S are defined respectively by

$$R = \frac{UL}{\nu},$$

$$S = \frac{B^2}{\mu \rho U^2},$$

which are called "Reynolds' number" and magnetic pressure number, respectively. The latter implies the ratio of the electromagnetic energy to the dynamical energy. Writing (16) in each component considering that the electric current density and the electric field have only the z component, we have

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \frac{1}{R}\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right) - S\,R_m\,(E_z' + u' B_y' - v' B_x')\,B_y', \tag{17}$$

$$\frac{\partial v'}{\partial t'} + u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} = -\frac{\partial p'}{\partial y'} + \frac{1}{R}\left(\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right) + S\,R_m\,(E_z' + u' B_y' - v' B_x')\,B_x'. \tag{18}$$

As considered in the previous section, when $R_m \ll 1$, the induced magnetic field is negligibly small, and we may only consider the applied magnetic field (for the magnetic field). But here we let $S\,R_m$ be of the order of unity to take the electromagnetic interactions into consideration. By this assumption, (17) and (18) are simplified.

In this paper, we investigate the electromagnetic interaction in the transverse magnetic field as in Rossow's investigations[1], in which the transverse magnetic field is applied in the y direction. Under the assumption $R_m \ll 1$, each component of the magnetic flux density is

$$B_x' \fallingdotseq O\,,$$
$$B_y' \fallingdotseq B_o',$$

where B_o' in the non-dimensional form is the applied magnetic field in the y direction.

Then, (17) and (18) become

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \frac{1}{R}\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right) - S\,R_m\,(E_z' + u' B_o')\,B_o'\,, \tag{19}$$

$$\frac{\partial v'}{\partial t'} + u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} = -\frac{\partial p'}{\partial y'} + \frac{1}{R}\left(\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right). \tag{20}$$

Now we approximate (19) and (20) by the ordinary boundary layer approximation. Performing the boundary layer approximation, we introduce the variables defined by

$$v^* = \sqrt{R}\;v',$$
$$y^* = \sqrt{R}\;y',$$

and put them into (19) and (20). Then, we have

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v^*\frac{\partial u'}{\partial y^*} = -\frac{\partial p'}{\partial x'} + \frac{1}{R}\left(\frac{\partial^2 u'}{\partial x'^2} + R\frac{\partial^2 u'}{\partial y^{*2}}\right) - S\,R_m\,(E_z' + u' B_o')\,B_o', \tag{21}$$

$$\frac{1}{\sqrt{R}}\frac{\partial v^*}{\partial t'}+\frac{1}{\sqrt{R}}u'\frac{\partial v^*}{\partial x'}+\frac{1}{\sqrt{R}}v^*\frac{\partial v^*}{\partial y^*}=-\sqrt{R}\frac{\partial p'}{\partial y^*}+\frac{1}{R}\left(\frac{1}{\sqrt{R}}\frac{\partial^2 v^*}{\partial x'^2}+\sqrt{R}\frac{\partial^2 v^*}{\partial y^{*2}}\right), \tag{22}$$

with the equation of continuity

$$\frac{\partial u'}{\partial x'}+\frac{\partial v^*}{\partial y^*}=O \tag{23}$$

Assuming that $\frac{1}{R} \ll 1$, we can obtain the boundary layer equations as follows:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0}{\rho}(E_z+uB_0), \tag{24}$$

$$\frac{\partial p}{\partial y}=O, \tag{25}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=O, \tag{26}$$

where we put them back the original dimensions. These equations are correct for the curved surface of the body as well as for the flat plate, if the curvature of the surface is of the order of $\frac{1}{\sqrt{R}}$.

5. On the Free Stream

In the previous section, we find that the pressure across the boundary layer does not change in the transverse magnetic field, when the induced magnetic field is assumed to be negligible. The pressure is a function of x only as in the ordinary hydrodynamical boundary layer. The pressure is presumed to be the value obtained from the solution of an inviscid fluid for the same body which exists approximately outside the boundary layer. Ordinary, the pressure is supposed to be known in a boundary layer problem. The v velocity component of the fluid outside the boundary layer is negligible, because the pressure across the boundary layer does not change. Therefore, we have

$$\frac{\partial u_\infty}{\partial t}+u_\infty\frac{\partial u_\infty}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}-\frac{\sigma B_0}{\rho}(E_z+u_\infty B_0), \tag{27}$$

where u_∞ is the velocity in the free stream parallel to the plate, or along the surface of the body outside the boundary layer, which is usually taken as the velocity for the corresponding inviscid solution. Putting (27) into (24), we have

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial u_\infty}{\partial t}+u_\infty\frac{\partial u_\infty}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}+\frac{\sigma B_0^2}{\rho}(u_\infty-u). \tag{28}$$

In the following sections, we investigate the unsteady boundary layer flow on the basis of (28), presuming that u_∞ (x, t) is the inviscid solution involved the electromagnetic effect under the suitable boundary conditions for the electromagnetic field.

6. Unsteady Boundary Layer Flow

The unsteady flow in the boundary layer may be divided into two classes: 1) the commencement of the boundary layer flow from rest, and 2) the unsteady flow for longer elapsed after the starting time. In the two dimensional incompressible flow, the former is investigated by Blasius[3], and Goldstein and Rosenhead[7], etc., the latter by Schlichting[8], and Lighthill[9], etc.. For the electrically conducting fluid, the latter case is investigated by the author[10] in the presence of the transverse magnetic field. In this paper, we investigate the former case ,that is, the commencement of the boundary layer flow along the surface of the body from rest. The equations governing this flow are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial u_\infty}{\partial t}+u_\infty\frac{\partial u_\infty}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}+\frac{\sigma B_0{}^2}{\rho}(u_\infty-u), \tag{28}$$

$$\frac{\partial p}{\partial y}=O, \tag{25}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=O, \tag{26}$$

with the boundary conditions

$$y=O\,;\quad u=v=O\,,\qquad y\to\infty\,;\ u\to u_\infty\,(x,\,t). \tag{29}$$

In the ordinary fluid, Blasius solved in the series expansions by the stream function ψ satisfied in the equation of continuity (26), as defined by

$$\frac{\partial\psi}{\partial y}=u\,, \tag{30}$$

$$\frac{\partial\psi}{\partial x}=-v\,, \tag{31}$$

using Rayleigh's solution[11] as the first approximation, which is justified in the investigation of the flow formation of Couette motion. The author have previously investigated the flow formation of Couette motion of the electrically conducting fluid in the transverse magnetic field[12]. So far as $N=SR_m$, the parameter which is the measure of the electromagnetic effect, is small, the velocity profile is very analogous to Rayleigh's case in the vicinity of the starting time. Therefore, we use the same series expansion method as Blasius' regarding Rayleigh's solution as the first approximation.

Let the flow of the fluid start to move at $t=0$. The initial conditions are

$$u_\infty = 0 \qquad t \leq 0 \ , \qquad u_\infty = u_\infty (x, t) \qquad t > 0 \ , \tag{32}$$

We define the stream function as

$$\psi (x, y, t) = 2 \sqrt{\nu t} \ u_\infty (x, t) \ \phi (x, \eta, t), \tag{33}$$

where

$$\eta = \frac{y}{2\sqrt{\nu t}} \ . \tag{34}$$

And hence,

$$u = u_\infty \frac{\partial \phi}{\partial \eta} \ , \tag{35}$$

$$v = - 2 \sqrt{\nu t} \left(\frac{\partial u_\infty}{\partial x} \phi + u_\infty \frac{\partial \phi}{\partial x} \right) . \tag{36}$$

Putting (35) and (36) into (28), we get

$$\frac{\partial^3 \phi}{\partial \eta^3} + 2\eta \frac{\partial^2 \phi}{\partial \eta^2} + 4 \frac{t}{u_\infty} \frac{\partial u_\infty}{\partial t} \left(1 - \frac{\partial \phi}{\partial \eta} \right) - 4t \frac{\partial^2 \phi}{\partial t \, \partial \eta} + 4t \frac{\sigma B_0{}^2}{\rho} \left(1 - \frac{\partial \phi}{\partial \eta} \right)$$
$$+ 4t \frac{\partial u_\infty}{\partial x} \left\{ 1 - \left(\frac{\partial \phi}{\partial \eta} \right)^2 + \phi \frac{\partial^2 \phi}{\partial \eta^2} \right\} + 4t \ u_\infty \left(\frac{\partial^2 \phi}{\partial \eta^2} \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial \eta} \frac{\partial^2 \phi}{\partial x \, \partial \eta} \right) = 0 \ . \tag{37}$$

And the boundary conditions are

$$\eta = 0 \ ; \ \phi = \frac{\partial \phi}{\partial \eta} = 0 , \quad \eta \to \infty \ ; \ \frac{\partial \phi}{\partial \eta} \to 1 \ . \tag{38}$$

The free stream $u_\infty (x, t)$ is generally assumed to be

$$u_\infty (x, t) = V (x) \ t^\alpha . \tag{39}$$

However, the condition of the existence of the solution is severe than in the ordinary boundary layer flow. When the applied magnetic field does not change in time, there are no solutions except $\alpha = 0$. When the magnetic field is varying according to $t^{-\frac{1}{2}}$, there are solutions for any values of α. For the latter, we will investigate in later papers. Here, we assume

$$\frac{\partial u_\infty}{\partial t} = 0 \ .$$

Now we expand $\phi (x, \eta, t)$ in ascending powers of t as

$$\phi (x, \eta, t) = \phi_0 (\eta) + t \left\{ \frac{d u_\infty}{dx} \phi_{11} (\eta) + m \ \phi_{12} (\eta) \right\}$$
$$+ t^2 \left\{ \left(\frac{d u_\infty}{dx} \right)^2 \phi_{21} (\eta) + u_\infty \frac{d^2 u_\infty}{dx^2} \phi_{22} (\eta) + m \frac{d u_\infty}{dx} \phi_{23} (\eta) + m^2 \phi_{24} (\eta) \right\} + \cdots\cdots, \tag{40}$$

where m is defined by

$$m = \frac{\sigma B_0^2}{\rho} .$$

Putting (40) into (37), the ordinary differential equations are obtained

$$\frac{d^3\phi_0}{d\eta^3} + 2\eta \frac{d^2\phi_0}{d\eta^2} = 0, \tag{41}$$

$$\frac{d^3\phi_{11}}{d\eta^3} + 2\eta \frac{d^2\phi_{11}}{d\eta^2} - 4\frac{d\phi_{11}}{d\eta} = -4\left\{1 - \left(\frac{d\phi_0}{d\eta}\right)^2 + \phi_0 \frac{d^2\phi_0}{d\eta^2}\right\}, \tag{42}$$

$$\frac{d^3\phi_{12}}{d\eta^3} + 2\eta \frac{d^2\phi_{12}}{d\eta^2} - 4\frac{d\phi_{12}}{d\eta} = -4\left(1 - \frac{d\phi_0}{d\eta}\right), \tag{43}$$

$$\frac{d^3\phi_{21}}{d\eta^3} + 2\eta \frac{d^2\phi_{21}}{d\eta^2} - 8\frac{d\phi_{21}}{d\eta} = -4\left(\phi_0\frac{d^2\phi_{11}}{d\eta^2} - 2\frac{d\phi_0}{d\eta}\frac{d\phi_{11}}{d\eta} + \frac{d^2\phi_0}{d\eta^2}\phi_{11}\right), \tag{44}$$

$$\frac{d^3\phi_{22}}{d\eta^3} + 2\eta \frac{d^2\phi_{22}}{d\eta^2} - 8\frac{d\phi_{22}}{d\eta} = -4\left(\frac{d^2\phi_0}{d\eta^2}\phi_{11} - \frac{d\phi_0}{d\eta}\frac{d\phi_{11}}{d\eta}\right), \tag{45}$$

$$\frac{d^3\phi_{23}}{d\eta^3} + 2\eta \frac{d^2\phi_{23}}{d\eta^2} - 8\frac{d\phi_{23}}{d\eta} = -4\left(\frac{d\phi_{11}}{d\eta} + \phi_0\frac{d^2\phi_{12}}{d\eta^2} - 2\frac{d\phi_0}{d\eta}\frac{d\phi_{12}}{d\eta} + \frac{d\phi^2_0}{d\eta^2}\phi_{12}\right), \tag{46}$$

$$\frac{d^3\phi_{24}}{d\eta^3} + 2\eta \frac{d^2\phi_{24}}{d\eta^2} - 8\frac{d\phi_{24}}{d\eta} = 4\frac{d\phi_{12}}{d\eta}, \tag{47}$$

with the boundary conditions

$$\phi_0(o) = \phi_{11}(o) = \phi_{12}(o) = \cdots\cdots = \phi_{24}(o) = O,$$

$$\frac{d}{d\eta}\phi_0(o) = O, \qquad \frac{d}{d\eta}\phi_0(\infty) = 1,$$

$$\frac{d}{d\eta}\phi_{11}(o) = \frac{d}{d\eta}\phi_{12}(o) = \cdots\cdots = \frac{d}{d\eta}\phi_{24}(o) = O,$$

$$\frac{d}{d\eta}\phi_{11}(\infty) = \frac{d}{d\eta}\phi_{12}(\infty) = \cdots\cdots = \frac{d}{d\eta}\phi_{24}(\infty) = O. \tag{48}$$

7. Solutions of the Differential Equations

The differential equations from (41) to (47) can be solved in terms of the parabolic cylinder function $D_n(\eta)$. Generally the differential equation

$$\frac{d^2F}{d\eta^2} + 2\eta\frac{dF}{d\eta} - 4\alpha F = O, \tag{49}$$

with the boundary condition

$$F(o) = 1, \qquad F(\infty) = O, \tag{50}$$

can be solved in terms of the particular parabolic cylinder function $g_\alpha(\eta)$ introduced by Watson[13] as

$$F(\eta) = 2^{2\alpha}\,\Gamma(\alpha+1)\,g_\alpha(\eta)\,. \tag{51}$$

Here, $g_\alpha(\eta)$ has the following properties

$$\frac{d}{d\eta}\,g_\alpha(\eta) = -\,g_{\alpha-\frac{1}{2}}(\eta), \tag{52}$$

$$g_0(\eta) = 1 - \operatorname{erf}\,\eta\,, \tag{53}$$

and

$$g_\alpha(o) = \frac{2^{-2\alpha}}{\Gamma(\alpha+1)}\quad, \tag{54}$$

where erf η is the error function, one of the parabolic cylinder functions, defined by

$$\operatorname{erf}\,\eta = \frac{2}{\sqrt{\pi}}\int_o^\eta e^{-\xi^2}\,d\xi\,. \tag{55}$$

Using these functions $g_\alpha(\eta)$, we can solve the differential equations from (41) to (47) with the boundary conditions (48).
From (41), we get

$$\frac{d\phi_0}{d\eta} = 1 - g_o\,, \tag{56}$$

which is the solution as the first approximation, and Rayleigh's solution. From (42) and (43), we get as the second approximation

$$\frac{d\phi_{11}}{d\eta} = 2g_0 + \frac{1}{4}\,g_{-1} - \frac{2}{3\sqrt{\pi}}\,g_{-\frac{1}{2}} + 2\left(g^2_{\frac{1}{2}} - g_0\,g_1\right) - 4\left(\frac{3}{2} + \frac{2}{3\sqrt{\pi}}\right)g_1\,, \tag{57}$$

$$\frac{d\phi_{12}}{d\eta} = g_0 - 4g_1\,. \tag{58}$$

(57) agrees with Blasius' result. The solution of (44) and (45) as the third approximation are complicated in the forms. They agree with Goldstein and Rosenhead's result[7]. The solutions of (46) and (47) are

$$\frac{d\phi_{23}}{d\eta} = -\frac{1}{4}\,g_{-1} + 4g_0\,g_1 - 12g_1^2 + \frac{4}{5\sqrt{\pi}}\,g_{-\frac{1}{2}} - \frac{8}{3\sqrt{\pi}}\,g_{-\frac{1}{2}} - 4g^2_{\frac{1}{2}} + 20\,g_{\frac{1}{2}}\,g_{\frac{3}{2}} - 8g_0\,g_2 - g_0 + 4\left(\frac{3}{2} + \frac{2}{3\pi}\right)g_1 - 32\left(\frac{1}{2} - \frac{6}{15\pi}\right)g_2, \tag{59}$$

$$\frac{d\phi_{24}}{d\eta} = -\frac{1}{2}\,g_0 + 4g_1 - 16g_2\,. \tag{60}$$

The skin friction from the second approximation is

$$\rho\nu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\rho\nu u_\infty}{\sqrt{\pi\nu t}}\left[1 + t\left\{\left(1 + \frac{4}{3\pi}\right)\frac{du_\infty}{dx} + m\right\}\right]. \tag{61}$$

The conditions of separation estimated from the second approximation is

$$\frac{d^2}{d\eta^2}\phi_0(o)+t\left\{\frac{du_\infty}{dx}\frac{d^2}{d\eta^2}\phi_{11}(o)+m\frac{d^2}{d\eta^2}\phi_{12}(o)\right\}=0\,. \tag{62}$$

And the lapsed time from the starting of the motion when separation begins is given by

$$1+t\left\{\left(1+\frac{4}{3\pi}\right)\frac{du_\infty}{dx}+m\right\}=0\,. \tag{63}$$

As above, we can solve the unsteady boundary layer flow of an electrically conducting fluid by the same method of the approximation as in the usually hydrodynamical boundary layer flows.

8. Unsteady Boundary Layer Flow on a Semi-infinite Flat Plate

Rossow investigated the steady boundary layer flow under the assumtion $R_m \ll 1$. When the plate is the perfect conductor, the z component of the electric field vanishes from (7). (27) becomes in the steady flow

$$u_\infty\frac{du_\infty}{dx}=-\frac{1}{\rho}\frac{\partial p}{\partial x}-mu_\infty\,. \tag{64}$$

On the flat plate, the pressure gracient vanishes. The free stream $u_\infty(x)$ is obtained approximately

$$u_\infty(x)=u_0\left(1-\frac{m}{u_0}x\right), \tag{65}$$

where u_0 is the velocity at the leading edge. We find that the free stream is the decelerating flow. Rossow analyzed this flow by the Howarth's method[14] for the decelerating flow in the boundary layer.

For the electrically conducting fluid in the presence of the transverse magnetic field, we can expect that there is something corresponding to the pressure gradient, even though the pressure gradient vanishes. Blasius' expansion methed does not hold for the ordinary unsteady boundary layer flow on a semi-infinite flat plate, but for the electrically conducting fluid Blasius' expansion method can be applied.

For the unsteady boundary layer flow of the electrically conducting fluid on a semi-infinite flat plate, we may replace in (40) by

$$\frac{du_\infty}{dx}=-m\,.$$

Or, we may expand the stream function as

$$\phi(x,\eta,t)=\phi_0(\eta)+mt\phi_1(\eta)+(mt)^2\phi_2(\eta)+\cdots\cdots. \tag{66}$$

The solutions can be obtained

the first approximation

$$\frac{d\phi_0}{d\eta} = 1 - g_0 , \tag{67}$$

the second approximation

$$\frac{d\phi_1}{d\eta} = -g_0 - \frac{1}{4} g_{-1} + \frac{2}{3\sqrt{\pi}} g_{-\frac{1}{2}} - 2 (g^2_{\frac{1}{2}} - g_0 g_1) + 4 \left(\frac{1}{2} + \frac{2}{3\pi} \right) g_1. \tag{68}$$

This result is very interesting in consideration of the existence of the solution for the two-dimensional slow motion by Stoke's approximation[15], for which it is not able to be applied in the ordinary hydrodynamics.

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電気的伝導性を有する流体の非定常境界層の流れ

片　桐　理　和

工学部　電気工学科

磁場が存在するもとでの電気的伝導性を有する流体の力学（電磁流体力学）は航宙科学の諸問題に多くの重要な基礎的解決を与えている。ミサイル，ロケット等の大気圏再突入の際の一つの減速手段として，電磁的効果を利用した境界層制御と呼ばれる手段があるが，理論的には，従来の流体力学で用いられているものと類似した方法で研究されている。本報告ではこの境界層制御に関して特に非定常な流れに対して，従来の流体力学で用いられたものと類似した方法により解析を行い，併せて電磁流体力学の理論的な面からの考察に関して，一つの指針を与えることを試みる。